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MEASUREMENT OF INHOMOGENEOUS MAGNETIC FIELDS USING A ROTATING DIPOLE SEARCH COIL

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Chemistry Division

September 1968

TABLE OF CONTENTS

LIST OF FIGURES

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ABSTRACT

The measurement of inhomogeneous magnetic fields using a rotating dipole search coil has been investigated for both two-dimensional fields and cylindrically symmetric three-dimensional fields. It is found that when the fields are expressed in terms of an appropriate series expansion, the coil shape may be determined such that the constant term of the expansion may be measured to any desired degree of accuracy. An optimum coil shape to minimize the contributions from terms of higher order is one with rectangular cross section and whose parameters satisfy the relation

$$\frac{9}{20}\,\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} - h^2 = 0,$$

where r_1 and r_2 are the outside and inside radii, and $2h$ is the full width. By use of such a coil with $x = r_1/r_2 = 0.3$, ignoring the fact that individual turns may not be flat and circular, but may actually be some sort of helical segment, and by ignoring possible deformations that may occur during the fabrication of the coil, except for a simple eccentricity of the rotation axis, the field of the Argonne double-focusing alpha-particle spectrometer may be measured with a theoretical precision of one part in 10^8. In actual practice, real imperfections in the coil plus other factors, such as the difficulty in measuring small voltages and the problems encountered when attempting to balance the phase of two similar signals, may limit the accuracy to about one part in 10^5.

I. INTRODUCTION

It is a natural consequence of Faraday's law of electromagnetic induction[1] that when a conductor moves through a magnetic field, an emf is induced along its length, and if the ends are joined by a closed circuit a current will flow. The induced emf is proportional to the velocity of motion and to that component of the magnetic field which is perpendicular to the direction of motion. That component of the field which is parallel to the direction of motion does not induce an emf in the conductor.

The rotating-coil gaussmeter is a simple application of this consequence of Faraday's law and may be used for precise measurements of magnetic fields.[1] When used for measuring a uniform magnetic field, the axis of rotation must be aligned perpendicular to the direction of the field, because the component of the field along the axis will not contribute to the measurement; the result of the measurement will then be the value of the field at the geometric center of the coil. In uniform fields the emf induced in the coil will be purely sinusoidal with the frequency of rotation of the coil; when measurements are made in nonuniform fields, emfs that are harmonics of the rotation frequency will be induced, and these harmonic components will add in such a way that the result of the measurement will not yield the value of the field at the geometric center of the coil. The effect of these harmonic components must be minimized in order that the measurement have any meaning. It is the purpose of this investigation to determine the magnitude of these harmonic contributions and to discuss means of minimizing or eliminating them altogether.

If the nonuniform field is expressed in terms of a Fourier expansion about the point (x, y, z) at which the measurement is to be made, the component of the field that we wish to measure is characterized by the constant term of the expansion. Then the contributions from the higher-order terms, which will correspond to derivatives of the field in an equivalent Taylor's expansion, must either vanish or be made negligibly small. This problem has been studied by Garrett[2] and by de Raad,[3] who have shown that in the measurement of two-dimensional fields, the dimensions of the probe coil may be determined such that the largest contribution to the measurement of the field will come from the fifth-order term, which involves the fourth derivative of the field, and is negligible. In the Garrett analysis, which has been discussed by Laslett,[4] the system is separated into two parts: the field region, containing no sources, and the source region, containing the current sources. This treatment is versatile and enables one to design, in principle, a current system to produce any arbitrary field configuration; conversely, the field configuration may be determined for any arbitrary source geometry. In this treatment, however, the physics of an extended dipole coil rotating in a magnetic field is obscured by the formalism, and it does not permit ready estimate of errors due to simple asymmetry in coil construction. It is also not readily extended to include the case of measurement of three-dimensional fields.

de Raad used a more direct approach to design a variety of coil systems for measurement of the various Fourier components of strongly inhomogeneous, two-dimensional fields. The general approach used by de Raad is followed in the present discussion and is straightforward.

The rotating-coil gaussmeters in common use today, such as those that are commercially available,[5] are similar to the two coil-balanced null, or the two coil-balanced minimum, instruments discussed in detail by

Hedgran[6] and by Bäckström.[7] The principle is very simple: the emf generated in the field-measuring coil is amplified by an amplifier tuned to the rotation frequency of the coil; this signal is balanced against the output from a reference coil rotating on the same shaft in a known reference field, and the difference signal is amplified and rectified for measurement and display.

The following discussion of the use of rotating dipole coils for measuring magnetic fields will be divided into two parts. The first part will be similar to the discussion by de Raad of the measurement of two-dimensional fields possessing a median plane of mirror symmetry. The present discussion will be extended to include the case for which the rotation axis of the probe coil lies in the median plane but may have any angular orientation with respect to the direction of the field gradient. The second part is a study of the measurement of three-dimensional fields with both cylindrical symmetry and a median plane of mirror symmetry. Again, the rotation axis lies in the median plane, but may have any arbitrary orientation with respect to the direction of the field gradient. The two-dimensional problem is a special case of the three-dimensional problem when the radius of curvature of the field region is so large with respect to the dimensions of the search coil that the segment of arc may be approximated by a straight line.

In this analysis we study an ideal extended dipole coil with rectangular cross section; each turn will be assumed to be a flat circular loop, and the coil will be assumed to be free of any distortion or irregularity in construction except for a simple eccentricity of the rotation axis with respect to the symmetry plane of the coil. The results obtained with such a coil will be presented in a general form, but will also be applied to the Argonne double-focusing alpha-particle spectrometer, which is described in Appendix I. With an ideal coil that possesses at most a 10% eccentricity of the rotation axis, the field may be measured to one part in 10^8. In reality, the coil will not be perfect, but will possess imperfections whose effects are difficult to assess: each turn is not flat and circular, but is some sort of helical segment; the rotation axis, in addition to a simple eccentricity, may suffer an arbitrary angular misalignment with respect to the symmetry plane; the finished coil may not be perfectly round and flat. In view of the manifold of possible imperfections, the accuracy will in all probability be somewhat less than one part in 10^8, and will more likely be closer to one part in 10^5 or 10^6, a figure which comes close to the limitations on the accuracy determined by factors other than errors in coil construction. Noise in the amplifier and slip-ring assembly limits the accuracy in the measurement of small voltages, and the difficulty in adjusting the relative phase of the field signal with respect to the reference signal in the two coil-balanced null or two coil-balanced minimum systems limits the precision of the measurement. An arbitrary extended dipole coil that is not designed to have the optimum shape is discussed in Appendix II, and it is seen that for measurements involving such weakly inhomogeneous magnetic fields as are encountered in the Argonne double-focusing alpha-particle spectrometer, the conditions to be satisfied by the coil parameters may be greatly relaxed.

II. EMF INDUCED IN A ROTATING COIL IN AN ARBITRARY MAGNETIC FIELD

A. Simple Rigid Loop Rotating in a Magnetic Field

When a simple circular loop of area a is placed in a magnetic field of flux density B such that the angle between the plane of the loop and the direction of the field is ψ, the flux linked by the coil is

$$\Phi = Ba \sin \psi.$$

If the flux linked by the coil changes with time, an emf

$$V = -\frac{d\Phi}{dt}$$

is induced in the loop. In the case of a rigid toroidal loop of constant area a rotating about a diameter perpendicular to the field direction, the angle ψ will be given by

$$\psi = \omega t,$$

where ω is the magnitude of the angular velocity. The emf induced in the loop will be

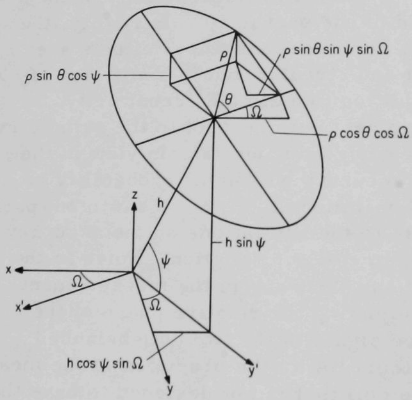

Fig. 1. Single Loop Rotating on the End of the Radius Arm of Length h, Showing the Relations between the Coordinates for the Two-dimensional Configuration

$$V = -\frac{d}{dt}(Ba \sin \omega t) = a\omega B \cos \omega t. \quad (1)$$

Consider a single loop at the end of a radius arm of length h, rotating about the x' axis, which lies in the x-y plane, and makes an angle Ω with the x axis (see Fig. 1). The plane of the loop is perpendicular to the radius arm. The coordinates of the loop are θ and ρ, and the element of area of the loop is

$$da = \rho d\rho d\theta$$

and the flux linked by the area element is

$$d\Phi = B \sin \psi\, \rho d\rho d\theta.$$

We assume that the field, B, is independent of time, perpendicular to the x-y plane, and expressed in terms of the variables ψ, h, ρ, θ, and Ω. The flux linked by the single loop will then be

$$\Phi_S = \sin\psi \int_0^{\rho} \rho d\rho \int_0^{2\pi} d\theta B \tag{2}$$

and the emf induced in the single loop as it rotates about the x' axis is

$$V_S = \omega \cos\psi \int_0^{\rho} \rho d\rho \int_0^{2\pi} d\theta B. \tag{3}$$

B. Thick Coil Rotating in a Magnetic Field

In practical measurements the coil will not consist of a single loop but consists of many turns of wire, and thus will have a finite extension in space. The results obtained with such an extended dipole coil will differ from those obtained with a single loop rotating about a diameter because in the extended coil with a plane of mirror symmetry only the turns that lie in the plane of symmetry will be rotating about a diameter. For our discussion we shall assume that the coil is that shown in Fig. 2. The inside radius is r_1, the outside radius is r_2, and the full width is 2h. We shall further assume that the wire diameter is d, and that successive layers of wire are wound with the wires side by side and directly on top of one another, as in Fig. 3. The areal density of turns will then be $1/d^2$, and the total number of turns in the coil will be

$$N = 2h(r_2 - r_1)/d^2. \tag{4}$$

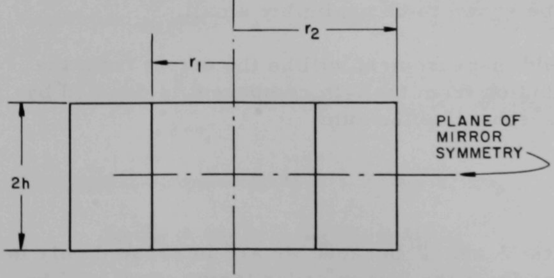

Fig. 2. Cross-sectional View of the Simple Extended Dipole Probe Coil

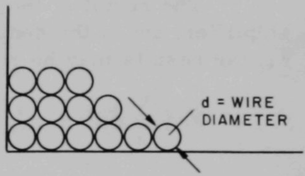

Fig. 3. Illustrating Perfect Windings, Each Turn Lying Directly on Top of the Corresponding Turn of the Preceding Layer

The emf induced in the entire extended coil is found by integrating (3) over ρ from r_1 to r_2, and over h from -h to +h:

$$V = \frac{1}{d^2}\int_{r_1}^{r_2} d\rho \int_{-h}^{h} dhV_s. \tag{5}$$

III. PROCEDURE AND NOTATION

When the field is expressed as a series expansion, and this expansion is substituted into Eq. (3), the emf induced around the single loop by the field component i will be denoted as V_{is}. The emf induced in the entire coil will be given by Eq. (5) and will be denoted as V_i. The total emf induced in the entire coil by all the field components will thus be the sum

$$V = \sum_i V_i.$$

In this analysis, the contribution from each component of the field will be calculated separately, and the coil parameters will then be determined such as to eliminate or minimize the contributions from the higher order terms in the expansion. If all of the contributions except that from the constant term can be eliminated, the measurement will yield the value of the field at the geometric center of the coil. Contributions from the higher order terms in the expansion will have the effect that the measurement will yield a value other than that at the center of the coil. Those contributions that cannot be eliminated by proper choice of coil parameters may be removed, or at least reduced, by passing the signal through a tuned, ac coupled amplifier which is tuned to the rotation frequency of the probe coil. The harmonic and dc components of the signal generated by the higher order terms are thus filtered out electronically, and the remaining contributions will be shown to be negligibly small.

The result of the field measurement will be the output from the amplifier, and if the contribution from the i-th component is denoted by $\mathcal{V}_i$, the results may be represented as the sum

$$\mathcal{V} = \sum_i \mathcal{V}_i.$$

We shall not give the results V and $\mathcal{V}$ because we are interested only in minimizing the contributions from the higher order terms, so it will be sufficient to calculate the individual contributions $\mathcal{V}_i$ in order to examine their relative effect on the measurement.

PART I: MEASUREMENT OF TWO-DIMENSIONAL MAGNETIC FIELDS

I. THE FOURIER EXPANSION OF THE TWO-DIMENSIONAL MAGNETIC FIELD

We shall consider the magnet configuration of Fig. 4, which has a plane of mirror symmetry, the x-y plane. In this median plane the field vector lies in the z direction, the field gradient lies along the x axis, and the field is constant in the y direction. The field at points off the median plane will have two components, B_x and B_z, both of which will contribute to a measurement made with a rotating-coil gaussmeter. A special case is encountered when the rotation axis lies along the x axis, and the component B_x will make no contribution. We shall consider the general case where the rotation axis will lie in the median plane but will be oriented at some arbitrary angle with the x axis. We shall calculate the contribution from each of the field components separately and determine the conditions under which the measurement will yield the value of B_z in the median plane to the desired accuracy, while simultaneously minimizing the contribution from B_x.

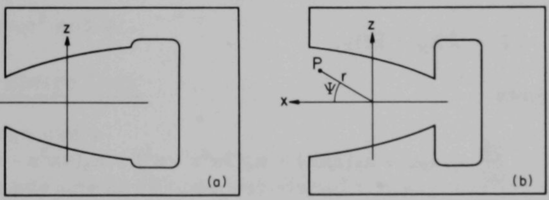

Fig. 4. Cross-sectional View of Two-dimensional Magnet Configurations Possessing Planes of Mirror Symmetry

The origin of the rectangular coordinate system shall be taken as shown in Fig. 4, and the magnetic scalar potential will be the solution of Laplace's equation in plane polar coordinates. A general solution is[8]

$$-\Phi = \sum_{k=0}^{\infty} (A_k r^k + B_k r^{-k})(C_k \sin k\Psi + D_k \cos k\Psi). \tag{6}$$

To avoid a singularity at $r = 0$, we set $B_k = 0$. If we then define

$$A_k C_k \equiv a_k/k; \qquad A_k D_k \equiv b_k/k, \tag{7}$$

the magnetic scalar potential may be written as

$$-\Phi = b_0 + \sum_{k=1}^{\infty} \frac{r^k}{k} (a_k \sin k\Psi + b_k \cos k\Psi), \tag{8}$$

where the a_k and b_k may be considered to be the coefficients of a Fourier expansion on the unit circle. By making use of the relations

$$x = r \cos \Psi; \qquad z = r \sin \Psi, \tag{9}$$

the scalar magnetic potential may be expressed in rectangular coordinates:

$$-\Phi = a_1 z + a_2 xz + a_3(x^2 z - \tfrac{1}{3} z^3) + a_4(x^3 z - xz^3) + \ldots$$

$$+ b_0 + b_1 x + \frac{b_2}{2}(x^2 - z^2) + \frac{b_3}{3}(x^3 - 3xz^2) + \ldots .$$

If we set $\Phi = 0$ in the x-y plane, all of the b_k vanish, and the plane $z = 0$ will be a plane of mirror symmetry. We shall have then

$$-\Phi = a_1 z + a_2 xz + a_3(x^2 z - \tfrac{1}{3} z^3) + a_4(x^3 z - xz^3) + a_5(x^4 z - 2x^2 z^3 + \tfrac{1}{5} z^5) + \ldots . \tag{10}$$

The magnetic field is a vector quantity

$$\vec{B} = -\nabla\Phi = \hat{x} B_x + \hat{z} B_z,$$

with components

$$B_x = -\frac{\partial \Phi}{\partial x} = a_2 z + a_3(2xz) + a_4(3x^2 z - z^3) + a_5(4x^3 z - 4xz^3) \tag{11}$$

and

$$B_z = -\frac{\partial \Phi}{\partial z} = a_1 + a_2 x + a_3(x^2 - z^2) + a_4(x^3 - 3xz^2) + a_5(x^4 - 6x^2 z^2 + z^4). \tag{12}$$

In the median plane $z = 0$, and we have

$$B_x = 0; \qquad B_z = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4, \tag{13}$$

where we see that the coefficients of the Fourier expansion in the median plane may be identified with the derivatives of B_z with respect to x in an equivalent Taylor's expansion of the field about the origin. The z field, B_z, consists of a constant, the uniform component, plus contributions from the various derivative terms, and the x field, B_x, consists only of contributions from the derivative terms. From Fig. 1 we see that components of the magnetic flux density linked by the area element may be expressed in terms of ψ, h, ρ, θ, and Ω by means of the relations

$$x = (h \cos \psi - \rho \sin \theta \sin \psi) \sin \Omega + \rho \cos \theta \cos \Omega; \tag{14a}$$

$$z = h \sin \psi + \rho \sin \theta \cos \psi. \tag{14b}$$

II. THE EMF INDUCED IN THE EXTENDED COIL BY THE z FIELD, B_z

We now express B in terms of the variables ψ, h, ρ, θ, and Ω by means of Eqs. (14a) and (14b), then making use of Eqs. (3) and (5), we calculate the emf induced in the coil by each component of the field. Owing to the mirror symmetry of the probe coil, the contributions from the odd derivative terms vanish. This follows from the fact that if we consider a Taylor's expansion of the field about the origin, then within a sufficiently small region about the origin the odd derivative terms will be linear, to a good approximation, and the integral of odd powers of h from -h to +h will vanish. The coil parameters r_1, r_2, and h, may then be determined such that the contribution from the second derivative term is zero also. The next largest contribution will come from the fourth derivative term which will be seen to be negligible.

A. The Uniform-field Component, a_1

1. The Single Loop

$$V_{1s} = a_1 \pi \omega \rho^2 \cos \psi. \tag{15}$$

2. The Entire Coil

$$V_1 = \tfrac{1}{2} V_0 a_1 \pi \cos \psi \tag{16}$$

where we have made use of (4) and introduced the expression

$$V_0 = \frac{2\omega N}{3}(r_2^2 + r_2 r_1 + r_1^2) = \frac{4h(r_2^3 - r_1^3)\,\omega}{3d^2}. \tag{17}$$

After full-wave rectification the output signal is

$$\mathcal{V}_1 = V_0 a_1. \tag{18}$$

B. The Gradient Component, a_2

1. The Single Loop

$$V_{2s} = a_2 \pi \omega h \rho^2 \sin \Omega \cos^2 \psi. \tag{18a}$$

2. The Entire Coil

$$V_2 = 0$$

because the integral over the odd powers of h from -h to +h will vanish.

C. The Second-derivative Component, a_3

1. The Single Loop

$$V_{3s} = a_3\pi\omega\{h^2\rho^2[\cos^3\psi \sin^2\Omega - \sin^2\psi \cos\psi] + \tfrac{1}{4}\rho^4[\sin^2\psi \cos\psi \sin^2\Omega - \cos^3\psi + \cos^2\Omega \cos\psi]\}. \tag{19}$$

2. The Entire Coil

$$V_3 = \frac{3}{4} V_0 a_3 \left[\frac{1}{20}\,\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} - \frac{1}{9} h^2\right](1 - 3\sin^2\Omega) \tag{20}$$

after full-wave rectification. If the coil is constructed so that the coil parameters satisfy the relation

$$\frac{9}{20}\,\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} - h^2 = 0, \tag{21}$$

then the second-derivative contribution will vanish. Equation (21) is the same condition found by Garrett[2] and de Raad.[3]

D. The Third-derivative Component, a_4

1. The Single Loop

$$V_{4s} = a_4\pi\omega[h^3\rho^2(\cos^4\psi \sin^3\Omega - 3\sin\psi \cos^3\psi \sin^3\psi) + \tfrac{3}{4}h\rho^4(\cos^2\psi \sin\Omega \cos^2\Omega + \sin^2\psi \cos^2\psi \sin^3\Omega - \sin\psi \cos\psi \cos^2\Omega - \sin^3\psi \cos\psi \sin^2\Omega - 2\sin\psi \cos^3\psi \sin^2\Omega)]. \tag{22}$$

2. The Entire Coil

$$V_4 = 0$$

because the integral over the odd powers of h from -h to +h vanishes.

E. The Fourth-derivative Component, a_5

1. The Single Loop

$$V_{5s} = a_5\pi\omega\{h^4\rho^2[\cos\psi \sin^4\psi - 6\sin^2\psi \cos^3\psi \sin^2\Omega + \cos^5\psi \sin^4\Omega] + \tfrac{3}{2}h^2\rho^4[\sin^2\psi \cos^3\psi + \sin^2\psi \cos\psi \cos^2\Omega + \cos^3\psi \sin^2\Omega \cos^2\Omega + (\sin^4\psi \cos\psi + \sin^2\psi \cos^3\psi - \cos^5\psi)\sin^2\Omega + \sin^2\psi \cos^3\psi \sin^4\Omega] + \tfrac{1}{8}\rho^6[\cos^5\psi - 2\cos^3\psi \cos^2\Omega - 6\sin^2\psi \cos^3\psi \sin^2\Omega + 2\sin^2\psi \cos\psi \sin^2\Omega \cos^2\Omega + \cos\psi \cos^4\Omega + \sin^4\psi \cos\psi \sin^4\psi]\}. \tag{23}$$

2. <u>The Entire Coil</u>

$$V_5 = \frac{3}{8}V_0 a_5\left[\frac{1}{56}\,\frac{r_2^7 - r_1^7}{r_2^3 - r_1^3}\,(13 - 18\,\sin^2\Omega + 5\,\sin^4\Omega) + \frac{h^2}{10}\,\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}(3\,\sin^2\Omega - 5\,\sin^4\Omega) + \frac{h^4}{15}(1 - 6\,\sin^2\Omega + 5\,\sin^4\Omega)\right] \tag{24}$$

after full-wave rectification and simplification of the Ω terms. Now making use of (21) and setting $x = r_1/r_2$, this expression becomes

$$V_5 = \frac{3}{8}V_0 a_5 r_2^4\left[\frac{1}{56}\,\frac{1 - x^7}{1 - x^3}\,(13 - 18\,\sin^2\Omega + 5\,\sin^4\Omega) + \frac{1}{500}\left(\frac{1 - x^5}{1 - x^3}\right)^2 (1 + 6\,\sin^2\Omega - 107.5\,\sin^4\Omega)\right], \tag{24a}$$

which will never be zero for an extended dipole coil, but if we take $x = r_1/r_2 = 0.3$, which corresponds to Garrett's 'H' coil, we shall have

$$V_5 < V_0 a_5\left(\frac{r_2^4}{10}\right) \tag{25}$$

for all Ω. The relative contribution of the fourth-derivative component to the field measurement will be

$$\frac{V_5}{V_1} < \frac{a_5}{a_1}\left(\frac{r_2^4}{10}\right) \tag{26}$$

This contribution may be made as small as we wish by using an appropriately small coil. Suppose that we are measuring the field in a typical alpha-particle spectrometer of the double-focusing type with a pole configuration similar to that of Fig. 6, see Appendix I for a description of the ANL alpha-particle spectrometer. We take the equilibrium orbit to be $x_0 = 30$ in. and the search coil dimensions to be of the order of 0.10 in. The three-dimensional field may then, without too great an error be approximated by our two-dimensional model (see Part II); the expansion coefficients of the two-dimensional model may be related to those of the three-dimensional model by comparing Eqs. (13) and (63), with the result that we simply replace a_n by a_n/x_0^{n-1}, so that we have $a_5/a_1 = 3.1 \times 10^{-7}/x_0^4 = 3.84 \times 10^{-13}$ and we have then

$$\frac{V_5}{V_1} \quad \frac{a_5}{a_1}\,10^{-5} < 10^{-17}, \tag{27}$$

which is negligibly small.

III. THE EMF INDUCED IN THE EXTENDED COIL BY THE x FIELD, B_x

The x component of the magnetic field differs significantly from the z component in the following ways:

1. It is parallel to the x-y plane.
2. There is no constant term in the Fourier expansion of the x field; the leading term is the first-derivative component.
3. The emf induced in the coil is 90° out of phase with the emf induced by the z field; it is a maximum for $\Omega = 90°$ and is zero for $\Omega = 0$ because no net emf is induced in the coil by fields parallel to the axis of rotation.

The contributions from the odd-derivative components will vanish as they did for the z field and only the even-derivative components will exist. The second-derivative term will vanish when condition (21) is satisfied, and the fourth-derivative contribution will be small. The analysis of the x field will be the same as the analysis of the z field in Sect. II. We shall make use of Eqs. (3), (5), and (11) through (14).

A. The Gradient Component, a_2

1. The Single Loop

$$V_{2s}^{x} = a_2 \pi \omega h \rho^2 \sin \psi \cos \psi . \tag{28}$$

2. The Entire Coil

$$V_2 = 0.$$

B. The Second-derivative Component, a_3

1. The Single Loop

$$V_{3s}^{x} = \tfrac{1}{2} a_3 \pi \omega (h^2 \rho^2 - \tfrac{1}{4}\rho^4) \sin \Omega \sin \psi \cos^2 \psi . \tag{29}$$

2. The Entire Coil

$$V_3^{x} = 0,$$

where we have made use of Eq. (21).

C. The Third-derivative Component, a_4

1. The Single Loop

$$V_{4s}^{x} = a_4\pi\omega[h^3\rho^2(3\sin\psi\cos^3\psi\sin^2\Omega - \sin^3\psi\cos\psi)$$
$$+ \tfrac{3}{4}h\rho^4(\sin\psi\cos\psi\cos^2\Omega + \sin^3\psi\cos\psi\sin^2\Omega$$
$$- \sin\psi\cos^3\psi(1 + 2\sin^2\Omega))]. \tag{30}$$

2. The Entire Coil

$$V_4^x = 0.$$

D. The Fourth-derivative Component, a_5

1. The Single Loop

$$V_{5s}^{x} = 4a_5\pi\omega\{h^4\rho^2[\sin\psi\cos^4\psi\sin^3\Omega - \sin^3\psi\cos^2\psi\sin\Omega]$$
$$+ \tfrac{3}{4}h^2\rho^2[\sin\psi\cos^2\psi\sin\Omega\cos^2\Omega$$
$$+ (\sin^3\psi\cos^2\psi - \sin\psi\cos^4\psi)(\sin^3\Omega + \sin\Omega)]$$
$$- \tfrac{1}{8}\rho^6[\sin\psi\cos^2\psi\sin\Omega\cos^2\Omega + \sin^3\psi\cos^2\psi\sin^3\Omega$$
$$+ \sin\psi\cos^4\psi\sin\Omega]\}. \tag{31}$$

2. The Entire Coil, after Full-wave Rectification

$$V_5^x = \frac{3}{2}V_0a_5\left[\frac{1}{56}\,\frac{r_2^7 - r_1^7}{r_2^3 - r_1^3}\,(1 - 2\sin\Omega + \sin^3\Omega)\right.$$
$$\left. + \frac{h^2}{10}\,\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\,(\sin\Omega - \sin^3\Omega) - \frac{h^4}{15}(1 - \sin^3\Omega)\right]. \tag{32}$$

If we use (21), set $x = r_1/r_2$, and rearrange the Ω terms, we get,

$$V_5^x = \frac{3}{2}V_0a_5r_2^4\left[\frac{1}{56}\,\frac{1 - x^7}{1 - x^3}\,(1 - 2\sin\Omega + \sin^3\Omega)\right.$$
$$\left. - \frac{9}{2000}\left(\frac{1 - x^5}{1 - x^3}\right)^2(3 - 10\sin\Omega + 9\sin^3\Omega)\right]. \tag{33}$$

Now if we take as typical values, $x = 0.3$ and $r_2 = 0.1$ in., we have

$$\mathcal{V}_5^x < V_0 a_5 \frac{10^{-4}}{70}, \tag{34}$$

and the relative magnitude of the contribution to the field measurement of the ANL alpha-particle spectrometer will be

$$\frac{\mathcal{V}_5^x}{\mathcal{V}_1} < \frac{a_5}{a_1}\,\frac{10^{-4}}{70} < 10^{-18}, \tag{35}$$

where again we have taken $a_5/a_1 = 3.84 \times 10^{-13}$.

IV. ESTIMATE OF ERRORS DUE TO PROBE-COIL ASYMMETRY

In this discussion of the effects of imperfect probe coil construction, we consider only the case where the rotation axis is offset from the plane of mirror symmetry, thus destroying the mirror symmetry of the coil. Errors in the winding itself will tend to be symmetrical about the mirror plane and will have the effect of rounding the corners by piling up extra turns near the center of the coil. Such effects, being symmetrical, will not introduce contributions from the otherwise absent components.

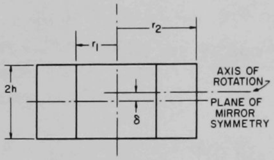

Fig. 5

The Simple Asymmetry with the Rotation Axis Displaced a Distance δ from the Plane of Mirror Symmetry

Suppose that the rotation axis is offset from the mirror plane by a small distance, δ, (see Fig. 5). Such a defect in coil construction will introduce contributions from the otherwise absent terms, and in this section we calculate the relative magnitude of these error terms. To do this we need only go back to (5) and change the limits of integration:

$$V = \frac{1}{d^2}\int_{r_1}^{r_2} d\rho \int_{-(h-\delta)}^{h+\delta} dh V_s. \tag{36}$$

The calculation of these errors is simplified because we can start with the expressions we have already derived for the emf induced in the single loop. The probe coil is no longer symmetrical, and now the integral of the odd powers of h will not vanish as they did in the above analysis. The odd derivatives will still be seen to vanish, however, when we simplify the ψ terms which reduce to constant terms and harmonics which are eliminated in the amplifier, so we consider only the contributions from the even-derivative components.

A. The z Field, B_z

1. The Uniform-field Component, (15)

$$V_1 = \frac{a_1 \pi \omega}{3d^2}(r_2^3 - r_1^3)[h + \delta + (h - \delta)] = \tfrac{1}{2} V_0 a_1 \pi \cos \psi \tag{37}$$

and, after full-wave rectification

$$\mathcal{V}_1 = V_0 a_1 \tag{38}$$

which is identical to (16), and the uniform component is not changed.

2. The Second-derivative Component, a_3

From Eq. (19) we have

$$\mathcal{V}_3' = \frac{3}{4} V_0 a_3 (1 - 3 \sin^2 \Omega) \left[\frac{1}{20} \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} - \frac{1}{9} h^2 - \frac{1}{3} \delta^2 \right], \tag{39}$$

which reduces to

$$\mathcal{V}_3' = \tfrac{1}{4} V_0 a_3 \delta^2 (3 \sin^2 \Omega - 1),$$

where we have made use of Eq. (21). If we assume that the eccentricity is not greater than 0.010 in., we will have

$$|\mathcal{V}_3'| \le V_0 a_3 (5 \cdot 10^{-5}) \tag{40}$$

and the relative contribution to the field measurement will be

$$\left| \frac{\mathcal{V}_3'}{\mathcal{V}_1} \right| \le \frac{a_3}{a_1} (5 \cdot 10^{-5}) \tag{41}$$

and if, as above, we relate the expansion coefficients of the two-dimensional model with the three-dimensional model,

$$\frac{a_3}{a_1} = \frac{2.67 \times 10^{-4}}{x_0^2} = 2.96 \times 10^{-7};$$

this becomes

$$\left| \frac{\mathcal{V}_3'}{\mathcal{V}_1} \right| < 10^{-10} \tag{42}$$

for the ANL alpha-particle spectrometer.

We see that the second-derivative component, which, by design, made no contribution to the field measurement for the case of a symmetric coil, does make a contribution if the coil is asymmetric. The effect is quite small, however: for example, if the eccentricity were equal to one-half width of the coil, we would have $\delta = h \sim 0.100$ in., and

$$\frac{V_3'}{V_1} < 10^{-8}.$$

The precision would still be limited by factors other than such errors due to coil asymmetry.

3. The Fourth-derivative Component, a_5

We have from Eq. (23)

$$V_5 = \frac{3}{8} V_0 a_5 \left[\frac{1}{56} \frac{r_2^7 - r_1^7}{r_2^3 - r_1^3} (13 - 18 \sin^2 \Omega + 5 \sin^4 \Omega) + \frac{1}{10} (h^2 + 3\delta^2) \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} (3 \sin^2 \Omega - 5 \sin^4 \Omega) + \frac{1}{15} (h^4 + 10h^2\delta^2 + 5\delta^4)(1 - 6 \sin^2 \Omega + 5 \sin^4 \Omega) \right]. \tag{43}$$

The eccentricity error is the difference between (43) and the contribution from the symmetric coil (24)

$$V_5' = \frac{3}{8} V_0 a_5 \left[\frac{3}{10} \delta^2 \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} (3 \sin^2 \Omega - 5 \sin^4 \Omega) + \frac{1}{3} (2h^2\delta^2 + \delta^4)(1 - 6 \sin^2 \Omega + 5 \sin^4 \Omega) \right], \tag{44}$$

and this reduces to

$$V_5' = \frac{3}{8} V_0 a_5 r_2^2 \delta^2 \left[\frac{3}{10} \frac{1 - x^5}{1 - x^3} (3 \sin^2 \Omega - 5 \sin^4 \Omega) + \frac{1}{3} \left(\frac{9}{10} \frac{1 - x^5}{1 - x^3} + \frac{\delta^2}{r_2^2} \right) (1 - 6 \sin^2 \Omega + 5 \sin^4 \Omega) \right]. \tag{45}$$

Taking the typical values from above

$$x = r_1/r_2 = 0.3; \qquad r_2 = 0.1 \text{ in.}; \qquad \delta = 0.01 \text{ in.},$$

we get an upper limit for the eccentricity error of

$$|V_5'| < V_0 a_5 (10^{-6}) \tag{46}$$

and the relative error is

$$\left|\frac{V_5'}{V_1}\right| < \frac{a_5}{a_1} 10^{-6} < 10^{-18}, \tag{47}$$

which is small compared with (27).

B. The x Field, B_x

1. The Second-derivative Component, a_3

From Eq. (29) we have

$$V_3^{x'} = \frac{1}{2} V_0 a_3 \sin \Omega \left[\frac{1}{20} \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} - \frac{1}{9} h^2 - \frac{1}{3} \delta^2\right] \tag{48}$$

which reduces to

$$V_3^{x'} = \tfrac{1}{2} V_0 a_3 \delta^2 \sin \Omega.$$

If we take $\delta = 0.01$ we have

$$|V_3^{x'}| \le V_0 a_3 (5 \cdot 10^{-5}), \tag{49}$$

and the relative contribution to the field measurement will be

$$\left|\frac{V_3^{x'}}{V_1}\right| \le \frac{a_3}{a_1} (5 \cdot 10^{-5}). \tag{50}$$

If we take $a_3/a_1 = 2.96 \times 10^{-7}$ as before,

$$\left|\frac{V_3^{x'}}{V_1}\right| < 10^{-10}, \tag{51}$$

and the same observations made following Eq. (42) will be valid here.

2. The Fourth-derivative Component, a_5

We have from Eq. (31)

$$V_5^x = \frac{3}{2} V_0 a_5 \left[\frac{1}{56} \frac{r_2^7 - r_1^7}{r_2^3 - r_1^3} (1 - 2 \sin \Omega + \sin^3 \Omega) + \frac{1}{10} \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} (h^2 + 3\delta^2)(\sin \Omega - \sin^3 \Omega) \right.$$
$$\left. + \frac{1}{15} (h^4 - 10h^2\delta^2 + 5\delta^4)(1 - \sin^3 \Omega)\right]. \tag{52}$$

The eccentricity error is the difference between (52) and (32), and reduces to

$$\mathcal{V}_5^{x'} = \frac{3}{2} V_0 a_5 r_2^2 \delta^2 \left[\frac{3}{10}\,\frac{1 - x^5}{1 - x^3}(\sin\Omega - \sin^3\Omega) + \frac{1}{3}\left(\frac{9}{10}\,\frac{1 - x^5}{1 - x^3} + \frac{\delta^2}{r_2^2}\right)(1 - \sin^3\Omega)\right]. \tag{53}$$

Again taking the typical values from above

$$x = r_1/r_2 = 0.3; \qquad r_2 = 0.1 \text{ in.}; \qquad \delta = 0.01 \text{ in.},$$

we get an upper limit for the eccentricity error

$$\mathcal{V}_5^{x'} < V_0 a_5\, 10^{-6}; \tag{54}$$

and the relative error

$$\frac{\mathcal{V}_5^{x'}}{\mathcal{V}_1} < \frac{a_5}{a_1}\, 10^{-6} < 10^{-18}, \tag{55}$$

is no larger than the contribution from the symmetrical coil, given by (35).

V. CONCLUSION

It is seen that for measuring the magnitude of a two-dimensional, inhomogeneous magnetic field with a rotating-coil gaussmeter, the optimum coil shape to minimize the contributions from the higher-order terms is one with a rectangular cross section and whose parameters satisfy Eq. (21)

$$h^2 - \frac{9}{20}\,\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} = 0,$$

which has been derived by Garrett[2] and by de Raad.[3] Using such a coil with $x = 0.3$, and possessing at most a 10% asymmetry of the rotation axis, the field of the Argonne double-focusing alpha-particle spectrometer may be measured with a theoretical precision of one part in 10^{10}, in the two-dimensional field approximation, independent of the relative orientation of the probe coil axis with respect to the direction of the field gradient. However, as was indicated in the Introduction, it is unlikely that a precision of greater than one part in 10^5 or 10^6 can be achieved in actual measurements, and as we saw above, even if the eccentricity was equal to half the width of the coil this precision would not be degraded. The theoretical results for the ANL spectrometer are summarized in Table I.

TABLE I

Relative Contribution	$\frac{V_2}{V_1}$	$\frac{V_3}{V_1}$	$\frac{V_4}{V_1}$	$\frac{V_5}{V_1}$
The z Component, B_z				
Symmetric coil	0	0	0	10^{-17}
Contribution from 10% eccentricity	0	10^{-10}	0	10^{-18}
The x Component, B_x				
Symmetric coil	0	0	0	10^{-18}
Contribution from 10% eccentricity	0	10^{-10}	0	10^{-18}

PART II: MEASUREMENT OF THREE-DIMENSIONAL MAGNETIC FIELDS

I. THE EXPANSION OF THE THREE-DIMENSIONAL MAGNETIC FIELD

Consider the magnetic field generated by either of the pole configurations of Fig. 6, both of which possess rotational symmetry. If we set the magnetic scalar potential, Φ, equal to zero for $z = 0$, then the x-y plane is a median plane of mirror symmetry, and owing to the rotational symmetry of the pole configuration the field will be cylindrically symmetric, that is, $\Phi = \Phi'(r,z) = \Phi(r',z)$, where r' is directed along r from x_0, see Fig. 7.

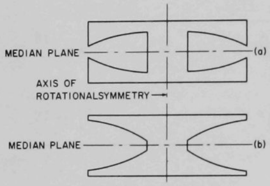

Fig. 6. Cross-sectional Views of Three-dimensional Magnet Configurations Possessing Both Cylindrical Symmetry and Median Planes of Mirror Symmetry

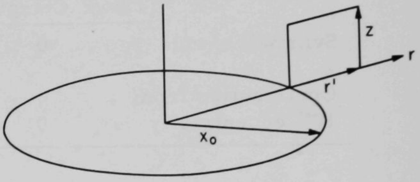

Fig. 7. The Coordinate System for the Cylindrically Symmetric, Three-dimensional Magnet Configuration

The scalar magnetic potential may then be written in expanded form about $r' = 0$, $z = 0$, as

$$-\Phi(r',z) = \left(A_{10} + A_{11}r' + A_{12}\frac{r'^2}{2!} + A_{13}\frac{r'^3}{3!} + \dots\right) z$$

$$+ \left(A_{30} + A_{31}r' + A_{32}\frac{r'^2}{2!} + \dots\right)\frac{z^3}{3!} + \left(A_{50} + A_{51}r' + \dots\right)\frac{z^5}{5!} + \dots$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{2m+1,n}\frac{r'^n}{n!}\frac{z^{2m+1}}{(2m+1)!}. \tag{56}$$

In this coordinate system, Laplace's equation becomes

$$\frac{1}{\left(1+\frac{r'}{x_0}\right)}\frac{\partial}{\partial r'}\left[\left(1+\frac{r'}{x_0}\right)\frac{\partial\Phi}{\partial r'}\right] + \frac{\partial^2\Phi}{\partial z^2} + \frac{1}{\left(1+\frac{r'}{x_0}\right)^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0. \tag{57}$$

Now $\Phi = \Phi(r',z)$, so this becomes

$$(x_0 + r')\left[\frac{\partial^2\Phi}{\partial r'^2} + \frac{\partial^2\Phi}{\partial z^2}\right] + \frac{\partial\Phi}{\partial r'} = 0. \tag{58}$$

From (56) and the fact that

$$\frac{\partial \Phi'}{\partial r} = \frac{\partial \Phi}{\partial r'},$$

it follows that

$$\frac{\partial \Phi}{\partial r'} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+1,n+1} \frac{r'^n}{n!} \frac{z^{2m+1}}{(2m+1)!};$$

$$\frac{\partial^2 \Phi}{\partial r'^2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+1,n+2} \frac{r'^n}{n!} \frac{z^{2m+1}}{(2m+1)!};$$

$$\frac{\partial^2 \Phi}{\partial z^2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+3,n} \frac{r'^n}{n!} \frac{z^{2m+1}}{(2m+1)!}. \tag{59}$$

Substitution into Laplace's Eq. (58) gives the recursion relations for all of the coefficients in terms of $A_{1,n}$:

$$A_{2m+3,n} = -\frac{1}{x_0}\left[(1+n)\, A_{2m+1,n+1} + rA_{2m+1,n+2} + nA_{2m+3,n-1}\right]. \tag{60}$$

The components of the magnetic flux density are

$$B_z = -\frac{\partial \Phi}{\partial z} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+1,n} \frac{r'^n}{n!} \frac{z^{2m}}{(2m)!}; \tag{61a}$$

$$B_{r'} = -\frac{\partial \Phi}{\partial r'} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+1,n+1} \frac{r'^n}{n!} \frac{z^{2m+1}}{(2m+1)!}; \tag{61b}$$

$$B_\varphi = 0. \tag{61c}$$

On the median plane of mirror symmetry, $z = 0$, and we have

$$B_z = \sum_{n=0}^{\infty} A_{1,n} \frac{r'^n}{n!};$$

$$B_{r'} = 0. \tag{62}$$

In Appendix I we have given the required field form for the ANL alpha-particle spectrometer in the form of a Taylor's expansion about the equilibrium orbit in the median plane.[9,10] Through fifth order terms it is given by

$$B_z = a_1 + a_2 \frac{r'}{x_0} + a_3\left(\frac{r'}{x_0}\right)^2 + a_4\left(\frac{r'}{x_0}\right)^3 + a_5\left(\frac{r'}{x_0}\right)^4, \tag{63}$$

where a_{n+1} is related to the n-th derivative of the z component of the field in the median plane on the equilibrium orbit:

$$a_{n+1} = \frac{1}{n!} \left.\frac{\partial^n B_z}{\partial r'^n}\right|_{r'=0} .$$

It follows from (62) and (63) that we must have

$$A_{10} = a_1; \quad A_{11} = a_2/x_0; \quad A_{12} = a_3(2!/x_0^2); \quad A_{13} = a_4(3!/x_0^3);$$
$$A_{14} = a_5(4!/x_0^4). \tag{64}$$

The magnetic flux density at all points in space will then be given by

$$\begin{aligned} B_z = a_1 &+ \frac{a_2}{x_0}\left[r' - \frac{z^2}{2x_0} + \frac{r'z^2}{2x_0} - \frac{3r'^2z^2}{4x_0^3} + \frac{z^4}{12x_0^3} - \frac{r'z^4}{12x_0^3}\right] \\ &+ \frac{a_3}{x_0^2}\left[r'^2 - z^2 - \frac{r'z^2}{x_0} + \frac{r'^2z^2}{x_0^2} - \frac{z^4}{12x_0^2}\right] \\ &+ \frac{a_4}{x_0^3}\left[r'^3 - 3r'z^2 - \frac{3r'^2z^2}{2x_0} + \frac{3r'^3z^2}{2x_0}\right] \\ &+ \frac{a_5}{x_0^4}\left[r'^4 - 6r'^2z^2 + z^4\right] \end{aligned} \tag{65}$$

and

$$\begin{aligned} B_{r'} = &\frac{a_2}{x_0}\left[z + \frac{z^2}{6x_0^2} - \frac{r'z^3}{2x_0^3}\right] + \frac{a_3}{x_0^2}\left[2r'z - \frac{z^3}{3x_0} + \frac{2r'z^3}{3x_0^2}\right] \\ &+ \frac{a_4}{x_0^3}\left[3r'^2z - z^3 - \frac{r'z^2}{x_0}\right] + \frac{a_5}{x_0^4}\left[4r'^3z - 4r'z^3\right] \end{aligned} \tag{66}$$

through terms of the order x_0^{-4}.

Suppose that we wish to measure the magnitude of such a magnetic field on the equilibrium orbit of a cyclical particle accelerator or charged particle spectrometer. Let x_0 be the radius of the equilibrium orbit, and choose a rectangular coordinate system such that the origin lies at the center of the orbit, and that the geometrical center of the probe coil is situated a distance x_0 from the origin along the X axis. The probe coil will be an extended dipole coil similar to that used in the investigation of the two-dimensional field, and the field will be sampled over the volume swept out by the coil. Let P be a point within this volume, and expand the scalar magnetic potential about a point that is located a distance x_0 from the origin along the radius vector r, which lies in the X-Y plane (see Fig. 8). Owing to the cylindrical symmetry of the field only the components B_r and B_z exist, and we shall relate them to the point x_0 along the X axis by expressing r' and z in terms of the coordinates of the coil.

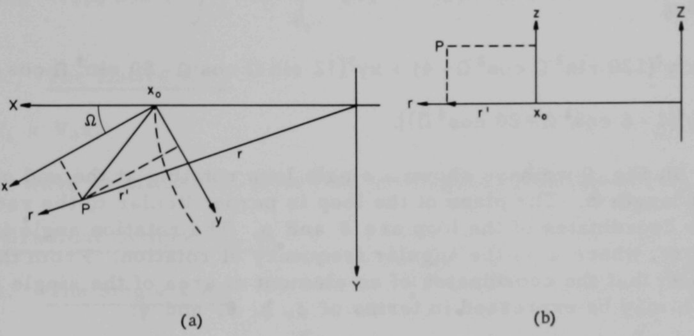

Fig. 8. (a) The X-Y Plane Showing the Coordinates for the Three-dimensional Configuration. (b) The r-Z Plane through the Point P.

II. THE COIL COORDINATES

We assume that the axis of rotation of the probe coil lies in the X-Y plane along the x axis, which makes an angle Ω with the X axis at the point x_0 (see Fig. 8). The coordinates (x, y, z) designate the point P within the volume swept out by the probe coil, and we have, from Fig. 8

$$r' = r - x_0, \tag{67}$$

where

$$r = \left[(x_0 + x \cos\Omega - y \sin\Omega)^2 + (x \sin\Omega + y \cos\Omega)^2\right]^{1/2} \tag{68}$$

or

$$r = x_0\left[1 + \frac{2}{x_0}(x \cos\Omega - y \sin\Omega) + \frac{x^2 + y^2}{x_0^2}\right]^{1/2} . \tag{69}$$

Now, if as in Part I, we take the probe coil dimensions to be of the order of 0.1 in., and the equilibrium orbit of a typical alpha-particle spectrometer as 30 in., we expand the radical and neglect terms of order higher than x_0^{-3}, and from (67) above we have

$$r' = x \cos \Omega - y \sin \Omega + \frac{1}{2x_0} [x^2 \sin^2 \Omega + 2xy \sin \Omega \cos \Omega + y^2 \cos^2 \Omega]$$

$$+ \frac{1}{2x_0^2} [x^3(\cos^3 \Omega - \cos \Omega) + x^2 y(\sin \Omega - 3 \sin \Omega \cos^2 \Omega)$$

$$- xy^2(\cos \Omega - 3 \sin^2 \Omega \cos \Omega) - y^3(\sin^3 \Omega - \sin \Omega)]$$

$$- \frac{1}{8x_0^3} [x^4(1 - 6 \cos^2 \Omega + 20 \cos^4 \Omega) + x^3 y(12 \sin \Omega \cos \Omega - 80 \sin \Omega \cos^3 \Omega)$$

$$+ x^2 y^2(120 \sin^2 \Omega \cos^2 \Omega - 4) + xy^3(12 \sin \Omega \cos \Omega - 80 \sin^3 \Omega \cos \Omega)$$

$$+ y^4(1 - 6 \cos^2 \Omega + 20 \cos^4 \Omega)]. \tag{70}$$

In Fig. 9 we have shown a single loop rotating at the end of a radius arm of length h. The plane of the loop is perpendicular to the radius arm and the coordinates of the loop are θ and ρ. The rotation angle is $\psi = \omega t$, as before, where ω is the angular frequency of rotation. From the figure it is seen that the coordinates of an element of area of the single loop (x, y, z), may be expressed in terms of ρ, h, θ, and ψ:

$$x = -\rho \cos \theta; \quad y = h \cos \psi - \rho \sin \theta \sin \psi; \quad z = h \sin \psi + \rho \sin \theta \cos \psi. \tag{71}$$

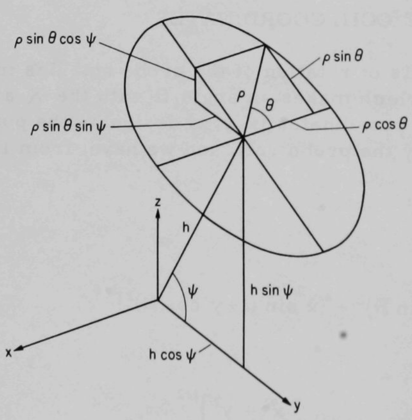

Fig. 9

The Single Loop Showing the Relations between the Coordinates for the Three-dimensional Configuration

Substitution of these equations into (65), (66), and (70) will give B_z and B_r in terms of the coil coordinates. Application of (3) and (5) will then give the emf induced in the coil by each component of the respective fields. The calculations will be carried out through terms of the order of x_0^{-4}.

III. THE EMF INDUCED IN THE EXTENDED COIL BY THE z FIELD, B_z

A. The Uniform-field Component, a_1

1. The Single Loop

$$V_{1s} = a_1\omega \cos\psi \int_0^{2\pi} d\theta \int_0^{\rho} \rho\, d\rho = a_1\omega\rho^2\pi \cos\psi. \tag{72}$$

2. The Entire Coil

$$\mathcal{V}_1 = V_0 a_1 \tag{73}$$

after full-wave rectification, where we have again made use of (17).

B. The Gradient Component, a_2

1. The Single Loop

$$V_{2s} = \frac{a_2\omega\cos\psi}{x_0}\int_0^{2\pi} d\theta \int_0^{\rho} \rho\, d\rho\left[r' - \frac{z^2}{2x_0} + \frac{r'z^2}{2x_0^2} - \frac{3r'^2z^2}{4x_0^3} + \frac{z^4}{12x_0^3} - \frac{r'z^4}{12x_0^3}\right]. \tag{74}$$

2. The Entire Coil

Retaining terms of the order of x_0^{-4}, we have for the entire coil after rectification

$$\begin{aligned}\mathcal{V}_2 = \frac{3V_0a_2}{8x_0^2}\Bigg\{&\frac{1}{9}\left[h^2 - \frac{9}{20}\,\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right](3\cos^2\Omega - 1)\\ &+ \frac{1}{8x_0^2}\Bigg[\frac{h^4}{15}\left(100\sin^4\Omega - 174\sin^2\Omega + \frac{233}{3}\right)\\ &+ \frac{h^2}{15}\,\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\left(-150\sin^4\Omega + 129\sin^2\Omega + \frac{35}{2}\right)\\ &+ \frac{1}{168}\,\frac{r_2^7 - r_1^7}{r_2^3 - r_1^3}(300\sin^4\Omega - 672\sin^2\Omega + 395)\Bigg]\Bigg\}.\end{aligned} \tag{75}$$

Now if we make use of (21) to eliminate the powers of h, the first term will vanish and we have

$$V_2 = \frac{3}{64} V_0 a_2 \left(\frac{r_2}{x_0}\right)^4 \left\{ \frac{3}{2000} \left(\frac{1 - x^5}{1 - x^3}\right)^2 (-2100 \sin^4 \Omega + 1024 \sin^2 \Omega + 1049) \right.$$

$$\left. + \frac{1}{168} \frac{1 - x^7}{1 - x^3} (300 \sin^4 \Omega - 672 \sin^2 \Omega + 395) \right\} \tag{76}$$

$$V_2 < \frac{3}{64} V_0 a_2 \left(\frac{r_2}{x_0}\right)^4 \left\{ 1.58 \left(\frac{1 - x^5}{1 - x^3}\right)^2 + 2.4 \frac{1 - x^7}{1 - x^3} \right\}. \tag{77}$$

We shall assume that we are to measure the field of the ANL alpha-particle spectrometer described in the Appendix, where we take

$x_0 = 30$ in.; $\quad r_2 = 0.1$ in.; $\quad x = 0.3$; $\quad a_2/a_1 = -1.67 \times 10^{-2}$.

The relative contribution of the gradient component to the field measurement will be

$$\left|\frac{V_2}{V_1}\right| < \left|\frac{a_2}{a_1}\right| 2.5 \cdot 10^{-11} < 10^{-12}. \tag{78}$$

C. The Second-derivative Component, a_3

1. The Single Loop

$$V_{3s} = \frac{a_3 \omega \cos \psi}{x_0^2} \int_0^{2\pi} d\theta \int_0^{\rho} \rho d\rho \left[r'^2 - z^2 - \frac{r' z^2}{x_0} + \frac{r'^2 z^2}{x_0^2} - \frac{z^4}{12 x_0^2} \right]. \tag{79}$$

2. The Entire Coil

$$V_3 = \frac{3 V_0 a_3}{4 x_0^2} \left\{ \frac{1}{9} \left[h^2 - \frac{9}{20} \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} \right] (3 \sin^2 \Omega - 1) \right.$$

$$+ \frac{1}{8 x_0^2} \left[\frac{h^4}{15} \left(5 \sin^4 \Omega + \sin^2 \Omega - \frac{7}{3} \right) \right.$$

$$+ \frac{h^2}{10} \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} (-25 \sin^4 \Omega + 23 \sin^2 \Omega + 1)$$

$$\left.\left. + \frac{1}{168} \frac{r_2^7 - r_1^7}{r_2^3 - r_1^3} (-25 \sin^4 \Omega + 27 \sin^2 \Omega + 39) \right] \right\}. \tag{80}$$

Again making use of (21) we have

$$V_3 = \frac{3}{32} V_0 a_3 \left(\frac{r_2}{x_0}\right)^4 \left\{ \frac{9}{2000} \left(\frac{1 - x^5}{1 - x^3}\right)^2 (-235 \sin^4 \Omega + 227 \sin^2 \Omega + 3) + \frac{1}{168} \frac{1 - x^7}{1 - x^3} (-25 \sin^4 \Omega + 27 \sin^2 \Omega + 39) \right\}. \tag{81}$$

It then follows that

$$V_3 < \frac{3}{32} V_0 a_3 \left(\frac{r_2}{x_0}\right)^4 \left[0.014 \left(\frac{1 - x^5}{1 - x^3}\right)^2 + 0.24 \frac{1 - x^7}{1 - x^3} \right], \tag{82}$$

and for the ANL alpha-particle spectrometer (see Appendix) we have for the relative contribution of the second-derivative component

$$\frac{V_3}{V_1} < \frac{a_3}{a_1} 3.1 \cdot 10^{-12} < 10^{-15}. \tag{83}$$

D. The Third-derivative Component, a_4

1. The Single Loop

$$V_{4S} = -\frac{a_4 \omega \pi \cos \psi}{x_0^3} \int_0^{2\pi} d\theta \int_0^{\rho} \rho d\rho \left[r'^3 - 3r'z^2 - \frac{3r'^2 z^2}{2x_0} + \frac{3r'^3 z^2}{2x_0} \right]. \tag{84}$$

2. The Entire Coil

$$V_4 = \frac{3V_0 a_4}{16 x_0^4} \left\{ \frac{h^4}{15} (-5 \sin^4 \Omega + 5 \sin^2 \Omega - 1) + \frac{h^2}{15} \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} (15 \sin^4 \Omega - 18 \sin^2 \Omega + 5) + \frac{1}{56} \frac{r_2^7 - r_1^7}{r_2^3 - r_1^3} (-15 \sin^4 \Omega + 21 \sin^2 \Omega - 7) \right\}. \tag{85}$$

Substituting from (21), we get

$$V_4 = \frac{3}{16} V_0 a_4 \left(\frac{r_2}{x_0}\right)^4 \left\{ \frac{3}{2000} \left(\frac{1 - x^5}{1 - x^3}\right)^2 (255 \sin^4 \Omega - 315 \sin^2 \Omega + 91) + \frac{1}{56} \frac{1 - x^7}{1 - x^3} (-15 \sin^4 \Omega + 21 \sin^2 \Omega - 7) \right\}. \tag{86}$$

$$V_4 < \frac{3}{16} V_0 a_4 \left(\frac{r_2}{x_0}\right)^4 \left[\frac{135}{2000}\left(\frac{1 - x^5}{1 - x^3}\right)^2 - \frac{1}{8}\,\frac{1 - x^7}{1 - x^3}\right]. \tag{87}$$

For the ANL spectrometer the relative contribution of the third-derivative component to the field measurement is

$$\left|\frac{V_4}{V_1}\right| < \left|\frac{a_4}{a_1}\right| 3 \cdot 10^{-12} < 10^{-17}. \tag{88}$$

E. The Fourth-derivative Component, a_5

1. The Single Loop

$$V_{5s} = \frac{a_5 \omega \cos \psi}{x_0^4} \int_0^{2\pi} d\theta \int_0^{\rho} \rho d\rho [r'^4 - 6r'^2 z^2 + z^4]. \tag{89}$$

2. The Entire Coil

$$V_5 = \frac{3V_0 a_5}{8x_0^4} \left\{\frac{h^4}{15}(5 \sin^4 \Omega - 6 \sin^2 \Omega + 1) + \frac{h^2}{10}\,\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}(-5 \sin^4 \Omega + 6 \sin^2 \Omega - 1)\right.$$

$$\left. + \frac{1}{168}\,\frac{r_2^7 - r_1^7}{r_2^3 - r_1^3}(120 \sin^4 \Omega - 202 \sin^2 \Omega + 187)\right\}. \tag{90}$$

Eliminate the powers of h with (21) and we have

$$V_5 = \frac{3}{8} V_0 a_5 \left(\frac{r_2}{x_0}\right)^4 \left\{\frac{9}{2000}\left(\frac{1 - x^5}{1 - x^3}\right)^2 (-35 \sin^4 \Omega + 42 \sin^2 \Omega - 7)\right.$$

$$\left. + \frac{1}{168}\,\frac{1 - x^7}{1 - x^3}(120 \sin^4 \Omega - 202 \sin^2 \Omega + 187)\right\}. \tag{91}$$

$$V_5 < \frac{3}{8} V_0 a_5 \left(\frac{r_2}{x_0}\right)^4 \left[-0.032\left(\frac{1 - x^5}{1 - x^3}\right)^2 + 1.11\,\frac{1 - x^7}{1 - x^3}\right]. \tag{92}$$

For the ANL alpha-particle spectrometer the relative contribution of the fourth-derivative component is

$$\frac{V_5}{V_1} < \frac{a_5}{a_1} 5.6 \cdot 10^{-11} < 10^{-16}. \tag{93}$$

IV. THE EMF INDUCED IN THE EXTENDED COIL BY THE RADIAL FIELD, B_r

A. The Gradient Component, a_2

1. The Single Loop

$$V_{2s}^{r} = \frac{a_2 \omega \cos \psi}{x_0} \int_0^{2\pi} d\theta \int_0^{\rho} \rho d\rho \left[z + \frac{z^2}{6x_0^2} - \frac{r' z^3}{2x_0^3} \right]. \tag{94}$$

2. The Entire Coil

After full-wave rectification,

$$\mathcal{V}_2^r = \frac{3}{8} V_0 a_2 \left\{ \frac{1}{x_0^3} \left[\frac{h^2}{27} + \frac{1}{20} \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} \right] + \frac{1}{x_0^4} \left[\frac{h^4}{30} - \frac{1}{112} \frac{r_2^7 - r_1^7}{r_2^3 - r_1^3} \right] \sin \Omega \right\}. \tag{95}$$

After eliminating the powers of h with (21), we shall have

$$\mathcal{V}_2^r < \frac{3}{8} V_0 a_2 \frac{r_2^2}{x_0^3} \left\{ \frac{1}{15} \frac{1 - x^5}{1 - x^3} + \frac{r_2^2}{x_0} \left[\frac{27}{4000} \left(\frac{1 - x^5}{1 - x^3} \right)^2 - \frac{1}{112} \frac{1 - x^7}{1 - x^3} \right] \right\}, \tag{96}$$

and for the ANL alpha-particle spectrometer the relative contribution of the gradient component to the field measurement will be

$$\left| \frac{\mathcal{V}_2^r}{\mathcal{V}_1} \right| < \left| \frac{a_2}{a_1} \right| 10^{-8} < 10^{-9}. \tag{97}$$

B. The Second-derivative Component, a_3

1. The Single Loop

$$V_{3s}^{r} = \frac{a_3 \omega \cos \psi}{x_0^2} \int_0^{2\pi} d\theta \int_0^{\rho} \rho d\rho \left[2r'z - \frac{z^3}{3x_0} + \frac{2r' z^3}{3x_0^2} \right]. \tag{98}$$

2. The Entire Coil

$$\mathcal{V}_3^r = \frac{3V_0 a_3}{2x_0^2} \left\{ \frac{1}{9} \left[h^2 - \frac{9}{20} \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} \right] \sin \Omega + \frac{1}{12x_0^2} \left[\frac{h^4}{15} (\sin \Omega - 3 \sin^3 \Omega) \right.\right.$$

$$\left.\left. + \frac{h^2}{20} \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} (9 \sin^3 \Omega - 7 \sin \Omega) + \frac{1}{56} \frac{r_2^7 - r_1^7}{r_2^3 - r_1^3} (3 \sin^3 \Omega - \sin \Omega) \right] \right\}. \tag{99}$$

Substituting from (21), we have

$$V_3^r = \frac{1}{8} V_0 a_3 \left(\frac{r_2}{x_0}\right)^4 \left[\frac{9}{2000}\left(\frac{1 - x^5}{1 - x^3}\right)^2 (36 \sin^3 \Omega - 32 \sin \Omega) + \frac{1}{56} \frac{1 - x^7}{1 - x^3} (3 \sin^3 \Omega - \sin \Omega)\right], \tag{100}$$

and we have

$$V_3^r < \frac{1}{8} V_0 a_3 \left(\frac{r_2}{x_0}\right)^4 \left[0.018 \left(\frac{1 - x^5}{1 - x^3}\right)^2 + 0.004 \frac{1 - x^7}{1 - x^3}\right]. \tag{101}$$

For the ANL spectrometer the relative contribution of the second-derivative component is

$$\left|\frac{V_3^r}{V_1}\right| < \left|\frac{a_3}{a_1}\right| 9 \cdot 10^{-13} < 10^{-15}. \tag{102}$$

C. The Third-derivative Component, a_4

1. The Single Loop

$$V_{4s} = \frac{a_4 \omega \cos \psi}{x_0^3} \int_0^{2\pi} d\theta \int_0^{\rho} \rho d\rho \left[3r'^2 z - z^3 - \frac{r' z^3}{x_0}\right]. \tag{103}$$

2. The Entire Coil

$$V_4^r = \frac{3V_0 a_4}{8x_0^4} \left\{\frac{h^4}{15}(3 \sin^3 \Omega - 2 \sin \Omega) - \frac{h^2}{10} \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}(3 \sin^3 \Omega - 2 \sin \Omega) + \frac{1}{112} \frac{r_2^7 - r_1^7}{r_2^3 - r_1^3}(2 \sin^3 \Omega - 3 \sin \Omega)\right\}. \tag{104}$$

Substituting from (21) we have

$$V_4^r = \frac{3}{8} V_0 a_4 \left(\frac{r_2}{x_0}\right)^4 \left\{- \frac{63}{2000}\left(\frac{1 - x^5}{1 - x^3}\right)^2 (3 \sin^3 \Omega - 2 \sin \Omega) + \frac{1}{112} \frac{1 - x^7}{1 - x^3} (2 \sin^3 \Omega - 3 \sin \Omega)\right\}; \tag{105}$$

$$\left|V_4^r\right| < \frac{3}{8}V_0a_4\left(\frac{r_2}{x_0}\right)^4\left[0.033\left(\frac{1-x^5}{1-x^3}\right)^2 + 0.01\,\frac{1-x^7}{1-x^3}\right]. \tag{106}$$

For the case of the ANL alpha-particle spectrometer, the relative contribution of the third-derivative component is

$$\left|\frac{V_4^r}{V_1}\right| < \left|\frac{a_4}{a_1}\right| 2.1\cdot 10^{-12} < 10^{-17}. \tag{107}$$

D. The Fourth-derivative Component, a_5

1. The Single Loop

$$V_{5s}^r = \frac{4a_5\omega\cos\psi}{x_0^4}\int_0^{2\pi} d\theta\int_0^{\rho}\rho\,d\rho\,(r'^3z - r'z^3). \tag{108}$$

2. The Entire Coil

$$V_5^r = \frac{3V_0a_5}{2x_0^4}\left\{\frac{h^4}{15} - \frac{h^2}{10}\left(\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right) + \frac{1}{56}\,\frac{r_2^7 - r_1^7}{r_2^3 - r_1^3}\right\}(\sin\Omega - \sin^3\Omega). \tag{109}$$

After substituting from (21), we have

$$V_5^r = \frac{3}{2}V_0a_5\left(\frac{r_2}{x_0}\right)^4\left[-\frac{63}{2000}\left(\frac{1-x^5}{1-x^3}\right)^2 + \frac{1}{56}\,\frac{1-x^7}{1-x^3}\right](\sin\Omega - \sin^3\Omega), \tag{110}$$

$$\left|V_5^r\right| < \frac{3}{2}V_0a_5\left(\frac{r_2}{x_0}\right)^4\left[-0.013\left(\frac{1-x^5}{1-x^3}\right)^2 + 0.007\,\frac{1-x^7}{1-x^3}\right]. \tag{111}$$

Then for the ANL alpha-particle spectrometer the relative contribution of the fourth-derivative component is

$$\left|\frac{V_5^r}{V_1}\right| < \frac{a_5}{a_1}1.1\cdot 10^{-12} < 10^{-18}. \tag{112}$$

V. ESTIMATE OF ERRORS DUE TO ASYMMETRY OF PROBE COIL

The same procedure will be followed as in Sect. IV of Part I. We shall assume that the rotation axis is offset from the plane of mirror symmetry by a small distance δ (see Fig. 5), and we integrate each component over the entire coil as per Eq. (36), starting with the expressions derived for the emf induced in the single loop.

A. The Emf Induced in the Asymmetric Coil by the z Field, B_z

1. The Uniform Component, a_1

$$V_1 = V_0 a_1,$$

which is identical to (73), and we see that the uniform component is unchanged; no error due to coil asymmetry is introduced.

2. The Gradient Component, a_2

We have from Eq. (74)

$$V_2 = \frac{3V_0 a_2}{8x_0^2}\left\{\frac{1}{9}\left[h^2 + 3\delta^2 - \frac{9}{20}\,\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right](3\cos^2\Omega - 1)\right.$$

$$+ \frac{1}{8x_0^2}\left[\frac{1}{15}(h^4 + 10h^2\delta^2 + 5\delta^4)\left(100\sin^4\Omega - 174\sin^2\Omega + \frac{233}{3}\right)\right.$$

$$+ \frac{1}{15}(h^2 + 3\delta^2)\,\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\left(-150\sin^4\Omega + 129\sin^2\Omega + \frac{35}{2}\right)$$

$$\left.\left. + \frac{1}{168}\,\frac{r_2^7 - r_1^7}{r_2^3 - r_1^3}(300\sin^4\Omega - 672\sin^2\Omega + 395)\right]\right\}. \tag{113}$$

The eccentricity error is the difference between (113) and (75), the symmetric relation, and after some simplification, we have

$$V_2' = \frac{3}{8}V_0 a_2\left(\frac{\delta}{x_0}\right)^2\left\{\frac{1}{3}(2 - 3\sin^2\Omega) + \frac{1}{8}\left(\frac{r_2}{x_0}\right)^2\left[\frac{1}{10}\,\frac{1 - x^5}{1 - x^3}(268 - 264\sin^2\Omega)\right.\right.$$

$$\left.\left. + \frac{1}{3}\left(\frac{\delta}{x_0}\right)^2(100\sin^4\Omega - 174\sin^2\Omega + 77.67)\right]\right\}; \tag{114}$$

$$V_2' < \frac{1}{4}V_0 a_2\left(\frac{\delta}{x_0}\right)^2\left\{1 + \frac{3}{16}\left(\frac{r_2}{x_0}\right)^2\left[26.8\,\frac{1 - x^5}{1 - x^3} + 25.9\left(\frac{\delta}{r_2}\right)^2\right]\right\}. \tag{115}$$

If, as in the case of the two-dimensional field analysis, we take $\delta = 0.010$ in., the relative contribution of the eccentricity error of the gradient component to the field measurement will be

$$\left|\frac{V_2'}{V_1}\right| < \left|\frac{a_2}{a_1}\right| 10^{-7} < 10^{-8} \tag{116}$$

for the case of the ANL spectrometer.

3. <u>The Second-derivative Component, a_3</u>

We have from Eq. (79)

$$V_3 = \frac{3V_0a_3}{4x_0^2}\left\{\frac{1}{9}\left[h^2 + 3\delta^2 - \frac{9}{20}\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right](3\sin^2\Omega - 1)\right.$$

$$+ \frac{1}{8x_0^2}\left[\frac{1}{15}(h^4 + 10h^2\delta^2 + 5\delta^4)\left(5\sin^4\Omega + \sin^2\Omega - \frac{7}{3}\right)\right.$$

$$+ \frac{1}{10}(h^2 + 3\delta^2)\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}(-25\sin^4\Omega + 23\sin^2\Omega + 1)$$

$$\left.\left. + \frac{1}{168}\frac{r_2^7 - r_1^7}{r_2^3 - r_1^3}(-25\sin^4\Omega + 27\sin^2\Omega + 39)\right]\right\}. \tag{117}$$

The eccentricity error is the difference between this expression and Eq. (80), and we have

$$V_3' = \frac{3}{4}V_0a_3\left(\frac{\delta}{x_0}\right)^2\left\{\frac{1}{3}(3\sin^2\Omega - 1) + \frac{1}{8}\left(\frac{r_2}{x_0}\right)^2\left[\frac{1}{10}\frac{1 - x^5}{1 - x^3}(-60\sin^4\Omega + 72\sin^2\Omega - 4)\right.\right.$$

$$\left.\left. + \frac{1}{3}\left(\frac{\delta}{r_2}\right)^2\left(5\sin^4\Omega + \sin^2\Omega - \frac{7}{3}\right)\right]\right\}; \tag{118}$$

$$V_3' \leq \frac{1}{2}V_0a_3\left(\frac{\delta}{x_0}\right)^2\left\{1 + \frac{3}{16}\left(\frac{r_2}{x_0}\right)^2\left[\frac{2}{5}\frac{1 - x^5}{1 - x^3} - \frac{7}{9}\left(\frac{\delta}{r_2}\right)^2\right]\right\}. \tag{119}$$

For the ANL alpha-particle spectrometer, the relative contribution of the eccentricity error of the second-derivative component will be

$$\frac{V_3'}{V_1} < \frac{a_3}{a_1}\,5.5 \cdot 10^{-8} < 10^{-10}. \tag{120}$$

4. <u>The Third-derivative Component, a_4</u>

We have from Eq. (84)

$$V_4 = \frac{3V_0a_4}{16x_0^4}\left\{\frac{1}{15}(h^4 + 10h^2\delta^2 + 5\delta^4)(-5\sin^4\Omega + 5\sin^2\Omega - 1)\right.$$

$$+ \frac{1}{15}(h^2 + 3\delta^2)\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}(15\sin^4\Omega - 18\sin^2\Omega + 5)$$

$$\left. + \frac{1}{56}\frac{r_2^7 - r_1^7}{r_2^3 - r_1^3}(-15\sin^4\Omega + 21\sin^2\Omega - 7)\right\}. \tag{121}$$

The eccentricity error is the difference between (121) and (85) and reduces to

$$V_4' = \frac{1}{16} V_0 a_4 \frac{r_2^2 \delta^2}{x_0^4} \left\{ \frac{3}{10} \frac{1 - x^5}{1 - x^3} (15 \sin^4 \Omega - 21 \sin^2 \Omega + 7) \right.$$

$$\left. + \left(\frac{\delta}{r_2}\right)^2 (-5 \sin^4 \Omega + 5 \sin^2 \Omega - 1) \right\}; \tag{122}$$

$$V_4' \leq \frac{1}{16} V_0 a_4 \frac{r_2^2 \delta^2}{x_0^4} \left[\frac{21}{10} \frac{1 - x^5}{1 - x^3} - \left(\frac{\delta}{r_2}\right)^2 \right]. \tag{123}$$

For the ANL spectrometer the relative contribution of the eccentricity error of the third-derivative component is

$$\left| \frac{V_4'}{V_1} \right| < \left| \frac{a_4}{a_1} \right| 2.5 \cdot 10^{-11} < 10^{-16}. \tag{124}$$

5. The Fourth-derivative Component, a_5

We have from Eq. (89)

$$V_5 = \frac{3 V_0 a_5}{8 x_0^4} \left\{ \frac{1}{15} (h^4 + 10 h^2 \delta^2 + 5 \delta^4)(5 \sin^4 \Omega - 6 \sin^2 \Omega + 1) \right.$$

$$+ \frac{1}{10} (h^2 + 3\delta^2) \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} (-5 \sin^4 \Omega + 6 \sin^2 \Omega - 1)$$

$$\left. + \frac{1}{168} \frac{r_2^7 - r_1^7}{r_2^3 - r_1^3} (120 \sin^4 \Omega - 202 \sin^2 \Omega + 187) \right\}. \tag{125}$$

The eccentricity error is the difference between (125) and (90), and is given by

$$V_5' = \frac{1}{8} V_0 a_5 \left(\frac{\delta}{x_0}\right)^4 (5 \sin^4 \Omega - 6 \sin^2 \Omega + 1); \tag{126}$$

$$|V_5'| < \frac{1}{8} V_0 a_5 \left(\frac{\delta}{x_0}\right)^4. \tag{127}$$

For the ANL spectrometer, the relative contribution of the eccentricity error of the fourth-derivative component is

$$\left| \frac{V_5'}{V_1} \right| < \frac{a_5}{a_1} 1.6 \cdot 10^{-15} < 10^{-21}. \tag{128}$$

B. The Emf Induced in the Asymmetric Coil by the Radial Field, B_r

1. The Gradient Component, a_2

We have from Eq. (94)

$$V_2^r = \frac{3}{8} V_0 a_2 \left\{ \frac{1}{x_0^3}\left[\frac{1}{27}(h^2+3\delta^2) + \frac{1}{20}\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right] + \frac{1}{x_0^4}\left[\frac{1}{30}(h^4+10h^2\delta^2+5\delta^4) - \frac{1}{112}\frac{r_2^7 - r_1^7}{r_2^3 - r_1^3}\right]\sin\Omega\right\}. \tag{129}$$

The eccentricity error is the difference between (129) and (95), and reduces to

$$V_2^{r'} \le \frac{1}{24} V_0 a_2 \frac{\delta^2}{x_0^3}\left\{1 + \frac{3}{2}\frac{r_2^2}{x_0}\left[\frac{9}{10}\frac{1-x^5}{1-x^3} + \left(\frac{\delta}{r_2}\right)^2\right]\right\}. \tag{130}$$

For the ANL spectrometer, the relative contribution of the eccentricity error of the gradient component to the field measurement will be

$$\left|\frac{V_2^{r'}}{V_1}\right| < \left|\frac{a_2}{a_1}\right| 1.6 \cdot 10^{-10} < 10^{-11}. \tag{131}$$

2. The Second-derivative Component, a_3

We have from Eq. (98)

$$V_3^r = \frac{3V_0a_3}{2x_0^2}\left\{\frac{1}{9}\left[h^2+3\delta^2 - \frac{9}{20}\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right]\sin\Omega + \frac{1}{12x_0^2}\left[\frac{1}{15}(h^4+10h^2\delta^2+5\delta^4)(\sin\Omega - 3\sin^3\Omega) + \frac{1}{20}(h^2+3\delta^2)\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}(9\sin^3\Omega - 7\sin\Omega) + \frac{1}{56}\frac{r_2^7 - r_1^7}{r_2^3 - r_1^3}(3\sin^3\Omega - \sin\Omega)\right]\right\}. \tag{132}$$

The eccentricity error is the difference between (132) and (99), and may be written as

$$V_3^{r'} = \frac{1}{2} V_0 a_3 \left(\frac{\delta}{x_0}\right)^2 \left\{ \sin\Omega + \frac{1}{4}\left(\frac{r_2}{x_0}\right)^2 \left[\frac{3}{20} \frac{1 - x^5}{1 - x^3} (6 \sin^3\Omega - 5 \sin\Omega) \right.\right.$$

$$\left.\left. + \left(\frac{\delta}{r_2}\right)^2 (\sin\Omega - 3 \sin^3\Omega) \right] \right\}; \qquad (133)$$

$$V_3^{r'} \leq \frac{1}{2} V_0 a_3 \left(\frac{\delta}{x_0}\right)^2 \left\{ 1 + \frac{1}{4}\left(\frac{r_2}{x_0}\right)^2 \left[\frac{3}{20} \frac{1 - x^5}{1 - x^3} - \left(\frac{\delta}{r_2}\right)^2 \right] \right\}. \qquad (134)$$

For the case of the ANL spectrometer, the relative contribution of the eccentricity error of the second-derivative component to the field measurement is

$$\frac{V_3^{r'}}{V_1} < \frac{a_3}{a_1} 5.5 \cdot 10^{-8} < 10^{-10}. \qquad (135)$$

3. <u>The Third-derivative Component, a_4</u>

We have from Eq. (103)

$$V_4^{r} = \frac{3V_0 a_4}{8x_0^4} \left\{ \frac{1}{15} (h^4 + 10h^2\delta^2 + 5\delta^4)(3 \sin^3\Omega - 2 \sin\Omega) \right.$$

$$- \frac{1}{10} (h^2 + 3\delta^2) \frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} (3 \sin^3\Omega - 2 \sin\Omega)$$

$$\left. + \frac{1}{112} \frac{r_2^7 - r_1^7}{r_2^3 - r_1^3} (2 \sin^3\Omega - 3 \sin\Omega) \right\}. \qquad (136)$$

The eccentricity error is the difference between (136) and (104), and reduces to

$$V_4^{r'} = \frac{1}{8} V_0 a_4 \left(\frac{\delta}{x_0}\right)^4 (3 \sin^3\Omega - 2 \sin\Omega); \qquad (137)$$

$$V_4^{r'} \leq \frac{1}{8} V_0 a_4 \left(\frac{\delta}{x_0}\right)^4. \qquad (138)$$

For the case of the ANL spectrometer, the relative contribution of the eccentricity error of the third-derivative component is

$$\left|\frac{V_4^{r'}}{V_1}\right| < \left|\frac{a_4}{a_1}\right| 1.56 \cdot 10^{-15} < 10^{-20}. \qquad (139)$$

4. The Fourth-derivative Component, a_5

We have from Eq. (108)

$$V_5^r = \frac{3V_0a_5}{2x_0^4}\left\{\frac{1}{15}(h^4+10h^2\delta^2+5\delta^4) - \frac{1}{10}(h^2+3\delta^2)\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right.$$
$$\left. + \frac{1}{56}\frac{r_2^7 - r_1^7}{r_2^3 - r_1^3}\right\}(\sin\Omega - \sin^3\Omega). \tag{140}$$

The eccentricity error is the difference between (140) and (109) and reduces to

$$V_5^{r'} = \frac{1}{2}V_0a_5\left(\frac{\delta}{x_0}\right)^4(\sin\Omega - \sin^3\Omega); \tag{141}$$

$$V_5^{r'} \leq 0.58V_0a_5\left(\frac{\delta}{x_0}\right)^4. \tag{142}$$

For the case of the ANL spectrometer, the relative contribution of the eccentricity error of the fourth-derivative component to the field measurement is

$$\frac{V_5^{r'}}{V_1} < \frac{a_5}{a_1}\,7.2\cdot 10^{-15} < 10^{-20}. \tag{143}$$

VI. CONCLUSION

It is seen that for measuring the magnitude of a three-dimensional, inhomogeneous magnetic field with a rotating-coil gaussmeter, the optimum coil shape to minimize the contributions from the higher-order components is one of rectangular cross section and whose parameters satisfy Eq. (21):

$$h^2 - \frac{9}{20}\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} = 0,$$

which is the same condition derived for the two-dimensional field model. Using such a coil with $x = 0.3$, and possessing at most a 10% asymmetry, the field of the Argonne double focusing alpha-particle spectrometer may be measured with a theoretical precision of one part in 10^8, independent of the relative orientation of the probe coil rotation axis with respect to the direction of the field gradient. When measurements are made at some radius other than the equilibrium orbit, that is, for an arbitrary radius $r = x_0 + x_1$, it can be seen that the contributions from the higher order terms will increase for $r > x_0$ and decrease for $r < x_0$. The relative contribution of the i-th order term will be changed by the factor $(1 + x_1/x_0)^{1-i}$, for we have from Appendix I

$$\frac{a_i}{a_1} \sim \frac{1}{x_0^{i-1}}$$

which will become

$$\frac{a_i}{a_1} \sim \frac{1}{(x_0 + x_1)^{i-1}} = \frac{1}{x_0^{i-1}}\left(1 + \frac{x_1}{x_0}\right)^{1-i} .$$

Over the range $r = \frac{1}{2}x_0$ to $r = 2x_0$, we see from Table II that the effect will be negligible for the case of the ANL spectrometer. Again, as was indicated in the Introduction, it is unlikely that in actual measurements the precision will be greater than one part in 10^5 or 10^6. If the eccentricity were equal to one half the width of the coil, that is, if we set $\delta = 0.100$ in. instead of 0.010 in., the largest contribution to the eccentricity error would be given by Eq. (116), and would be of the order of one part in 10^6. Thus we see that even a very large asymmetry would have little effect on the precision of an actual measurement. The theoretical results for the ANL spectrometer are summarized in Table II.

TABLE II

Relative Contribution	$\frac{V_2}{V_1}$	$\frac{V_3}{V_1}$	$\frac{V_4}{V_1}$	$\frac{V_5}{V_1}$
The z Component, B_z				
Symmetric coil	10^{-12}	10^{-15}	10^{-17}	10^{-16}
Contribution from 10% eccentricity	10^{-8}	10^{-10}	10^{-16}	10^{-21}
The Radial Component, B_r				
Symmetric coil	10^{-9}	10^{-15}	10^{-17}	10^{-18}
Contribution from 10% eccentricity	10^{-11}	10^{-10}	10^{-20}	10^{-20}

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APPENDIX I

The Argonne Alpha-particle Spectrometer

The Argonne alpha-particle spectrometer is a double-focusing instrument with an equilibrium orbit of $x_0 = 30$ in. The pole faces have been designed so that the z component of the magnetic field in the median plane satisfies the Taylor's expansion about the equilibrium orbit (see Fig. 8):

$$B_z = B_0\left[1 - \eta\,\frac{r - x_0}{x_0} + \beta\left(\frac{r - x_0}{x_0}\right)^2 + \gamma\left(\frac{r - x_0}{x_0}\right)^3 + \delta_1\left(\frac{r - x_0}{x_0}\right)^4 + \delta_2\left(\frac{r - x_0}{x_0}\right)^5 + \delta_3\left(\frac{r - x_0}{x_0}\right)^6\right], \tag{I-1}$$

where

$$\eta = 0.5; \quad \beta = 0.24; \quad \gamma = -0.065;$$
$$\delta_1 = 0.252; \quad \delta_2 = -0.70; \quad \delta_3 = -1.90, \tag{I-2}$$

and B_0 is the magnitude of the magnetic field on the median plane at $r = x_0$, the equilibrium orbit. Using the above values, we can calculate the ratios of the expansion coefficients in Eq. (63):

$$a_1 = B_0; \tag{I-3}$$

$$a_2 = \left.\frac{\partial B}{\partial r}\right|_{r=x_0} = -\eta\,\frac{B_0}{x_0}; \quad \frac{a_2}{a_1} = -\frac{\eta}{x_0} = -1.67 \cdot 10^{-2}; \tag{I-4}$$

$$a_3 = \frac{1}{2}\left.\frac{\partial^2 B_z}{\partial r^2}\right|_{r=x_0} = \beta\,\frac{B_0}{x_0^2}; \quad \frac{a_3}{a_1} = \frac{\beta}{x_0^2} = 2.67 \cdot 10^{-4}; \tag{I-5}$$

$$a_4 = \frac{1}{6}\left.\frac{\partial^3 B_z}{\partial r^3}\right|_{r=x_0} = \gamma\,\frac{B_0}{x_0^3}; \quad \frac{a_4}{a_1} = \frac{\gamma}{x_0^3} = -2.4 \cdot 10^{-6}; \tag{I-6}$$

$$a_5 = \frac{1}{24}\left.\frac{\partial^4 B_z}{\partial r^4}\right|_{r=x_0} = \delta_1\,\frac{B_0}{x_0^4}; \quad \frac{a_5}{a_1} = \frac{\delta_1}{x_0^4} = 3.1 \cdot 10^{-7}. \tag{I-7}$$

APPENDIX II

The Nonoptimum Coil Shape

Let us consider the effect of using an extended dipole coil whose parameters do not satisfy Eq. (21). Take a coil with[5] $r_2 = 0.155$ in., $r_1 = 0.053$ in., and full width $2h = 0.106$ in., so that $x = r_1/r_2 = 0.3$. This coil is constructed so that the full width is just one-half that required by condition (21) and is an extended dipole coil with square cross section. We shall calculate the leading terms for the two- and three-dimensional models.

I. The Two-dimensional Model

A. The z Field, B_z

For the symmetric coil, condition (21) is now not satisfied, so that the contribution from the second-derivative component will not vanish, and we have from (20)

$$\frac{V_3}{V_1} < \frac{a_3}{a_1}\frac{r_2^2}{6}\left\{\frac{h^2}{r_2^2} - \frac{9}{20}\frac{1-x^5}{1-x^3}\right\}; \tag{II-1}$$

or

$$\frac{V_3}{V_1} < \frac{a_3}{a_1} 3.2 \cdot 10^{-5} < 10^{-11}. \tag{II-2}$$

For the asymmetric coil, we shall again assume that the asymmetry takes the form of a 10% eccentricity, that is, the rotation axis is offset from the plane of mirror symmetry by a distance $\delta = 0.01$ in. (see Fig. 5). We then have from (39)

$$\left|\frac{V_3'}{V_1}\right| < \frac{a_3}{a_1}\frac{r_2^2}{6}\left\{-\left[\frac{h^2}{r_2^2} - \frac{9}{20}\frac{1-x^5}{1-x^3}\right] + 27\left(\frac{\delta}{r_2}\right)^2\right\} \tag{II-3}$$

or

$$\left|\frac{V_3'}{V_1}\right| < \frac{a_3}{a_1} 4.2 \cdot 10^{-4} < 10^{-9}. \tag{II-4}$$

B. The x Field, B_x

From Eqs. (39) and (48) we see that the same expressions are obtained for the x field as for the z field, and the above relations hold for this case also.

II. The Three-dimensional Model

A. The z Field, B_z

For the symmetric coil, we have from (75) for the leading terms involving the gradient component, a_2,

$$\left|\frac{V_2}{V_1}\right| \le \frac{1}{12}\left|\frac{a_2}{a_1}\right|\left(\frac{r_2}{x_0}\right)^2\left[\frac{9}{20}\,\frac{1 - x^5}{1 - x^3} - \frac{h^2}{r_2^2}\right]; \tag{II-5}$$

$$\left|\frac{V_2}{V_1}\right| < \left|\frac{a_2}{a_1}\right| 2 \cdot 10^{-6} < 10^{-7}. \tag{II-6}$$

For the second-derivative component, a_3, we have from (80) for the leading terms

$$\left|\frac{V_3}{V_1}\right| < \frac{1}{6}\,\frac{a_3}{a_1}\left(\frac{r_2}{x_0}\right)^2\left[\frac{9}{20}\,\frac{1 - x^5}{1 - x^3} - \frac{h^2}{r_2^2}\right] \tag{II-7}$$

$$< \frac{a_3}{a_1}\, 3.6 \cdot 10^{-6} < 10^{-9}. \tag{II-7a}$$

For the asymmetric coil we shall again assume the eccentricity to be of the order of 10%. The eccentricity errors will be seen to be of the same order of magnitude as those associated with a coil that satisfies Eq. (21), and will be smaller than the relative contribution from the corresponding components when using a symmetric coil. For the gradient component, a_2, we have from (115) for the leading term

$$\left|\frac{V_2'}{V_1}\right| < \frac{1}{4}\left|\frac{a_2}{a_1}\right|\left(\frac{\delta}{x_0}\right)^2; \tag{II-8}$$

$$\left|\frac{V_2'}{V_1}\right| < \left|\frac{a_2}{a_1}\right| 10^{-7} < 10^{-8}. \tag{II-9}$$

For the second-derivative component, a_3, we have from (119) for the leading term

$$\frac{V_3'}{V_1} < \frac{1}{2}\,\frac{a_3}{a_1}\left(\frac{\delta}{x_0}\right)^2; \tag{II-10}$$

$$\frac{V_3'}{V_1} < \frac{a_3}{a_1}\, 5.5 \cdot 10^{-8} < 10^{-10}. \tag{II-11}$$

B. The Radial Field, B_r

For the symmetric coil, we have from (95) for the leading term involving the gradient component, a_2,

$$\left|\frac{V_2^r}{V_1}\right| \leq \frac{1}{72}\left|\frac{a_2}{a_1}\right| \frac{r_2^2}{x_0^3}\left[\frac{h^2}{r_2^2} + \frac{27}{20}\,\frac{1 - x^5}{1 - x^3}\right]; \tag{II-12}$$

$$\left|\frac{V_2^r}{V_1}\right| < \left|\frac{a_2}{a_1}\right| 1.4 \cdot 10^{-6} < 10^{-7}. \tag{II-13}$$

From (99) for the second-derivative component, a_3, the leading terms are

$$\left|\frac{V_3^r}{V_1}\right| \leq \frac{1}{6}\,\frac{a_3}{a_1}\left(\frac{r_2}{x_0}\right)^2\left[\frac{9}{20}\,\frac{1 - x^5}{1 - x^3} - \frac{h^2}{r_2^2}\right]; \tag{II-14}$$

$$\left|\frac{V_3^r}{V_1}\right| < \frac{a_3}{a_1}\, 3.6 \cdot 10^{-6} < 10^{-9}. \tag{II-15}$$

For the asymmetric coil, we have from (130) that the leading term involving the gradient component, a_2, is

$$\left|\frac{V_2^{r'}}{V_1}\right| \leq \frac{1}{24}\left|\frac{a_2}{a_1}\right| \frac{\delta^2}{x_0^3}, \tag{II-16}$$

$$\left|\frac{V_2^{r'}}{V_1}\right| < \left|\frac{a_2}{a_1}\right| 1.55 \cdot 10^{-10} < 10^{-11}. \tag{II-17}$$

From (134) the leading term involving the second-derivative component, a_3, is

$$\left|\frac{V_3^{r'}}{V_1}\right| < \frac{1}{2}\,\frac{a_3}{a_1}\left(\frac{\delta}{x_0}\right)^2; \tag{II-18}$$

$$\frac{V_3^{r'}}{V_1} < \frac{a_3}{a_1}\, 5.55 \cdot 10^{-8} < 10^{-10}. \tag{II-19}$$

For the case of this particular coil, the theoretical precision is not determined by the 10% eccentricity, but by the coil parameters and the failure to satisfy Eq. (21). It is apparent that the real limit on the precision of the measurement is still determined by factors other than imperfections in coil construction, and will be of the order of 10^5 or 10^6. In general, it is seen that when making measurements on such weakly inhomogeneous magnetic fields as are generated in the ANL spectrometer, any conditions to be satisfied by the coil parameters may be greatly relaxed.

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X